Solution Sheet 11

Exercise 11.1

Prove that a sequence of probability measures (μ_n) on $C([0,T];\mathbb{R})$ is tight if and only if both

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}}\mu_n\left(\{f : |f(0)| > \lambda\}\right) = 0 \tag{1}$$

and for all $\varepsilon > 0$,

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\mu_n\left(\left\{f : \max_{|t-s|<\delta}|f(t)-f(s)| > \varepsilon\right\}\right) = 0. \tag{2}$$

Hint: Recall the following version of Arzelá-Ascoli: The closure of a set $A \subset C([0,T];\mathbb{R})$ is relatively compact if and only if both

$$\sup_{f\in A}|f(0)| < \infty$$

and

$$\lim_{\delta\to 0}\sup_{f\in A}\max_{|t-s|<\delta}|f(t)-f(s)| = 0.$$

Proof.

$\Longrightarrow$: Assume that (μ_n) is tight, thus for every $\kappa > 0$ there exists a compact $K \subset C([0,T];\mathbb{R})$ such that $\mu_n(K) > 1-\kappa$ for all $n \in \mathbb{N}$. From the hint we have that for sufficiently large λ and for all $f \in K$, $|f(0)| < \lambda$. Therefore

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}}\mu_n\left(\{f : |f(0)| > \lambda\}\right) \leq \sup_{n\in\mathbb{N}}\mu_n\left(K^C\right) < \kappa$$

and as $\kappa > 0$ was arbitrary, (1) is shown. Condition (2) follows similarly from the second part of the hint, as for all sufficiently small δ, all $f \in K$ and any given $\varepsilon > 0$, then

$$\max_{|t-s|<\delta}|f(t)-f(s)| \leq \varepsilon$$

so

$$\lim_{\delta\to 0}\sup_{n\in\mathbb{N}}\mu_n\left(\left\{f : \max_{|t-s|<\delta}|f(t)-f(s)| > \varepsilon\right\}\right) \leq \sup_{n\in\mathbb{N}}\mu_n\left(K^C\right) < \kappa.$$

$\Longleftarrow$: Assume (1) and (2). Fix $\kappa > 0$, with the intention of finding a compact K such that $\mu_n(K) > 1-\kappa$ for all $n \in \mathbb{N}$. From (1) we choose a λ such that

$$\sup_{n\in\mathbb{N}}\mu_n\left(\{f : |f(0)| > \lambda\}\right) \leq \frac{\kappa}{2}$$

and from (2) a sequence (δ_j) approaching zero such that

$$\sup_{n\in\mathbb{N}}\mu_n\left(\left\{f : \max_{|t-s|<\delta_j}|f(t)-f(s)| > \frac{1}{j}\right\}\right) \leq \frac{\kappa}{2^{j+1}}.$$

Now we define

$$K = \{f : |f(0)| \leq \lambda\} \cap \bigcap_{j=1}^{\infty}\left\{f : \max_{|t-s|<\delta_j}|f(t)-f(s)| \leq \frac{1}{j}\right\}.$$

K is a countable intersection of closed sets hence it is closed, and by the hint it is compact. In addition, $\mu_n(K^C) \leq \sum_{j=1}^{\infty}\frac{\kappa}{2^j} = \kappa$, as required.

□

Exercise 11.2

Define $\mathscr{F}$ to be the collection of step functions with compact support in $\mathbb{R}_+$, that is, of functions f which can be written

$$f = \sum_{j=1}^{n} \lambda_j \mathbb{1}_{(t_{j-1}, t_j]}$$

for constants λ_j. Write

$$\mathscr{E}^f := \int_0^T f(s) dW_s.$$

Prove that the set $\left\{\mathscr{E}^f : f \in \mathscr{F}\right\}$ is total in $L^2\left(\Omega, \mathcal{F}_T, \mathbb{P}\right)$ (that is, its span is dense).

Proof. See *Continuous Martingales and Brownian Motion, Revuz and Yor*, Page 198 Lemma 3.1. □

Exercise 11.3

For any give $F \in L^2\left(\Omega, \mathcal{F}_T, \mathbb{P}\right)$, prove that there exits a unique predictable process H such that

$$F = \mathbb{E}(F) + \int_0^T H_s dW_s.$$

Proof. See *Continuous Martingales and Brownian Motion, Revuz and Yor*, Page 199 Proposition 3.2. □

Exercise 11.4

Let $T : \mathcal{X} \to \mathcal{X}$ be a measurable mapping, and $\xi : \Omega \to \mathcal{X}$ satisfy that $T\xi = \xi$ in law. Show that for every measurable function f, the discret time process $(f(T^n\xi))_n$ is stationary.

Proof. Observe that

$$f(T^{n+1}\xi) = f(T^n(T\xi)) = f(T^n\xi) = f(\xi)$$

where equality holds in law due to the assumption.

□